An Introduction to Complex Numbers

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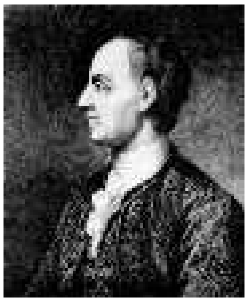
1 Introduction

There are equations like $ax^2 + bx + c = 0$ $(a, b \in \mathbb{R}, a \neq 0)$ which do not have a solution in $\mathbb{R}$. When we came over equations like $7x = 3$ in $\mathbb{Z}$, we had to introduce the rational numbers $\mathbb{Q}$ to get a solution. For similar reasons we had to introduce the irrational numbers $\mathbb{R}$ to get a solution for $x^2 = 7$ in $\mathbb{Q}$. Is there something equivalent with the quadratical equations in $\mathbb{R}$? The answer is yes, let's introduce Complex Numbers ...

Figure 1: Gauß (left) and Euler (right)[1]



One of the first persons being conscious of this problem was Leonhard Euler (1707-1783). He introduced a new "number" called the imaginary unit i with $i^2 = -1$. A few years later, Carl Friedrich Gauß (1777-1855) constructed the field of complex numbers $(\mathbb{C}, +, *)$.

2 Definitions

There are several ways to define and note down complex numbers. One of them is to write a complex number z as $a + ib$ with $a, b \in \mathbb{R}$, another would be to write

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it as a tuple $z = (a, b)$. We will stay with the first way of notation. Therefore we have to introduce imaginary numbers of the form ib, with $b \in \mathbb{R}, i^2 = -1$. Now, $x^2 = -r$ has solutions $x_{1/2} = \pm i\sqrt{r}$. For the general quadratic equation, the solutions are of the form $x_{1/2} = a \pm ib, (a, b \in \mathbb{R})$. We call $\bar{z} = a - bi$ the complex conjugate of $z = a + ib$.
In general, the solutions of the quadratic equation $ax^2 + bx + c = 0$ are

$$x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Every quadratic equation has two solutions in $\mathbb{C}$.
For example, the complex solutions of $x^2 + 2x + 3 = 0$ are

$$x_{1/2} = \frac{-2 \pm \sqrt{4 - 12}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{2}i$$

3 A very special "proof"[2]

You might stop here, thinking that the sole knowledge of the existence of complex numbers is already sufficient. To encourage you to read through all the following definitions and understand them, look at the following proof (especially at the last line). If you do not try to understand more about complex numbers, you will have to believe that $\pi = 0$.

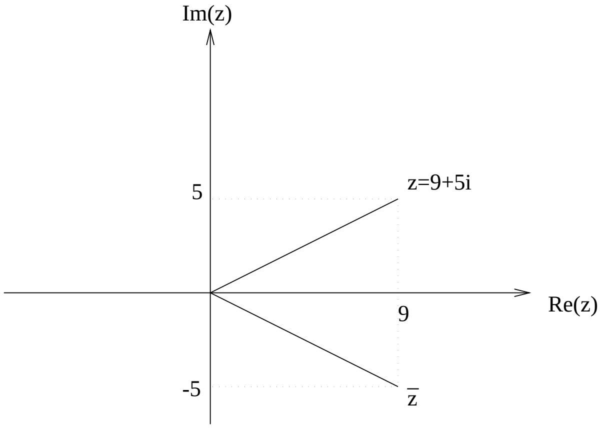
$$\begin{aligned} e^{2\pi i} &= 1 && (1)\\ \Rightarrow e^{2\pi i+1} &= e && (2)\\ \Rightarrow (e^{2\pi i+1})^{2\pi i+1} &= e && (3)\\ \Rightarrow e^{-4\pi^2} e^{4\pi i} e &= e && (4)\\ \Rightarrow e^{-4\pi^2} &= 1 && (5)\\ \Rightarrow \pi &= 0 && (6) \end{aligned}$$

As you might guess, there is something wrong about it. We will examine this later when we know more about the rules.
Note: There is more to it than just $e^{2\pi i} = e^0 \Rightarrow 2\pi i = 0$. This conclusion cannot be drawn since the exponential function in $\mathbb{C}$ is not injective [3].

4 Geometrical representation

Since we usually write complex numbers divided into two parts - the real and the imaginary one - it makes sense to look at a complex number geometrically in two dimensions. There is a convention to use the x-axis for the real and the y-axis for the imaginary part. This representation is called Argand Diagram (see Fig.2), named after Jean Robert Argand (1768-1822).

Figure 2: Argand Diagram for $z = 9 + 5i$ and $\bar{z} = 9 - 5i$



5 General calculus

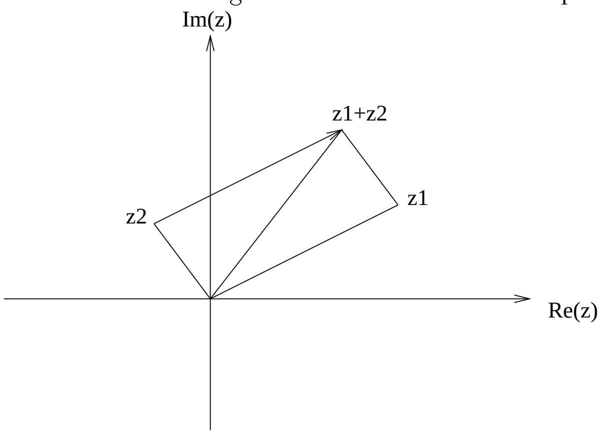
5.1 Addition

Complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are added by adding the real and the imaginary part separately.

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Example: $(3 + 2i) + (4 + 17i) = 7 + 19i$
We can compare the addition of two complex numbers with a vector addition in $\mathbb{R}^2$.

Figure 3: Addition of two complex numbers



5.2 Subtraction

The subtraction works similar to the addition:

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

It can be compared with a vector subtraction in $\mathbb{R}^2$.

5.3 Multiplication

To multiply two complex numbers, we multiply each part of them and separate the real and the imaginary part afterwards.

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

If we multiply a complex number $z = x + iy$ with its complex conjugated $z = x - iy$, we get:

$$z\bar{z} = x^2 + y^2 = |z|^2$$

Multiplying a complex number z by a real number r can be seen as a scalar multiplication for vectors, which means a rescaling of the vector. Multiplying a complex number z by the imaginary unit i can be seen as a $90°$ counterclockwise rotation.

$$z * i = (x + iy) * i = (-y + ix)$$

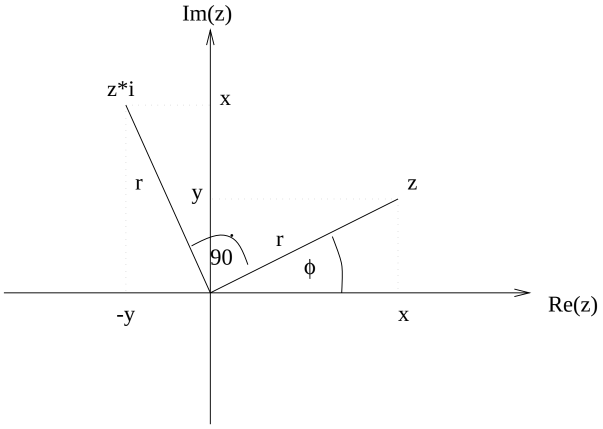


Figure 4: Multiplication by i

5.4 Division

To divide a complex number $z_1 = x_1 + iy_1$ by another complex number $z_2 = x_2 + iy_2$, it is necessary to make the denominator a real number. Therefore we have

$$z = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} * \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_1^2 + x_2^2}$$

$$\Rightarrow z = \frac{x_1 x_2 + y_1 y_2}{x_1^2 + x_2^2} + i \frac{y_1 x_2 - x_1 y_2}{x_1^2 + x_2^2}$$

6 Euler's formula

A very useful way to express trigonometric equations is to write them as

$$e^{iy} = \cos y + i \sin y,$$

and therefore

$$e^{x+iy} = e^x (\cos y + i \sin y)$$

The complex numbers e^{iy} are all laying on the unit circle. Now we can certify (1). $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + 0i = 1$. Euler's formula is especially useful for solving second order differential equations (see 5. in Section 11).

7 Polar Coordinates

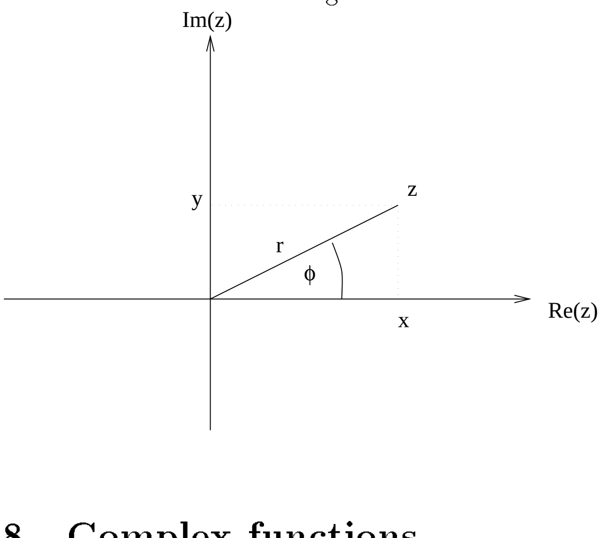
Another way to describe complex numbers is by using polar coordinates instead of the cartesian coordinates we used before. For a given complex number $z \in \mathbb{C}; z = x + iy; x, y \in \mathbb{R}$, z is written as

$$z = r(\cos\phi + i \sin\phi) = re^{i\phi}$$

with

$$r = \sqrt{x^2 + y^2}; \phi = \arctan\frac{y}{x}$$

Figure 5: Polar coordinates



8 Complex functions

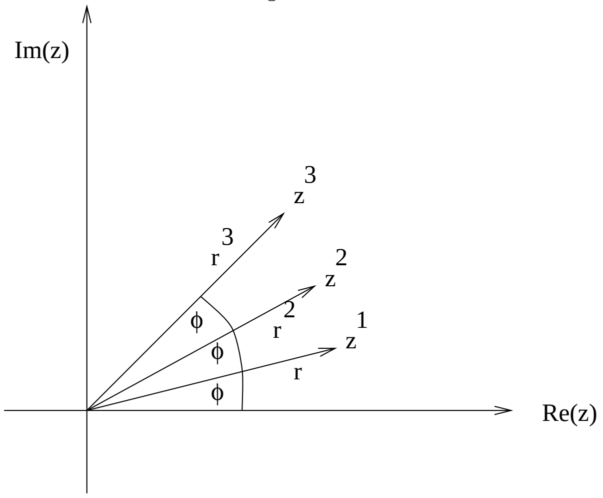
8.1 The power of

Taking a complex number to a power of something is the same as repeatedly multiplying it with itself. In geometrical terms, it is a repetition of turning and

scaling.
A more general formula in polar coordinates is called DeMoivre's equation:

$$z^n = (re^{i\phi})^n = r^n e^{in\phi}$$

Figure 6: z to the power of 1, 2 and 3



8.2 The exponential function

The exponential function $exp : \mathbb{C} \to \mathbb{C}$ is defined as:

$$\exp(x + iy) := e^x (\cos y + i \sin y)$$

As a result from this definition we get:

$$e^{i\phi} = e^{i(\phi + 2\pi n)}, n \in \mathbb{Z} \tag{7}$$

8.3 The logarithm

A set of solutions for z is needed for the equation $e^z = u; u, z \in \mathbb{C}, u \neq 0$

$$u = e^z = e^{x+iy} = e^x e^{iy} \tag{8}$$

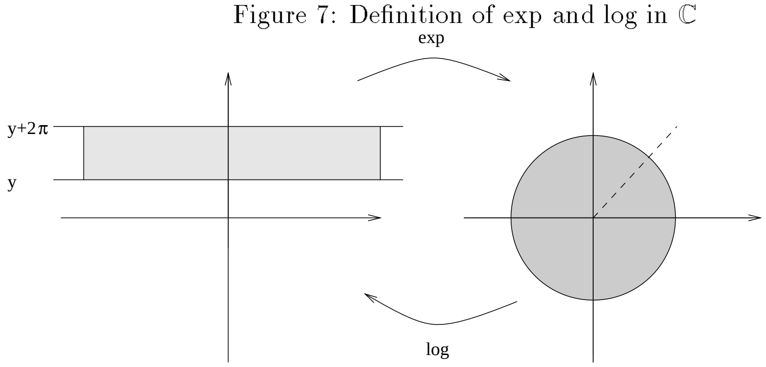
$$\Rightarrow |u| = e^x$$

$$(8) \Rightarrow u = |u| e^{iy}$$

$$\Rightarrow e^{iy} = \frac{u}{|u|} \tag{9}$$

If we try to take the logarithm from the last equation, we soon realize, that the logarithm has an infinite number of solutions! For $y \in [0, 2\pi)$ there is exactly one solution y_0, however, for other values of y, the solutions are of the form $y_0 + 2\pi n; n \in \mathbb{Z}$.

Figure 7: Definition of exp and log in $\mathbb{C}$



Since the exponential function is a many-to-one function (see (7)), its inverse, the logarithm, is multi-valued. The logarithm in $[0, 2\pi)$ is called the main branch of the logarithm.
As a second result from (9), we can see again, that complex numbers of the form e^{iy} are all laying on the unit circle.

9 Back to our proof

Now that we know a bit more about some essential operations and functions in $\mathbb{C}$, we could glance back at the "proof" (Section 3).
Let us have a look at (3). We can write it as

$$e^{(2\pi i+1)\log e^{2\pi i+1}}$$

where $\log e^{2\pi i+1} = 1$ when we consider the main branch of the logarithm. We have to use the main branch of the logarithm, since we started the "proof" with Euler's formula $e^{2\pi i} = 1$ as an assumption. It would be fatal to think that the rule

$$a^{b^c} = a^{bc}$$

can also be used for complex numbers since the complex logarithm is multi-valued not giving unique solutions. The thinking mistake can therefore be found in (4).

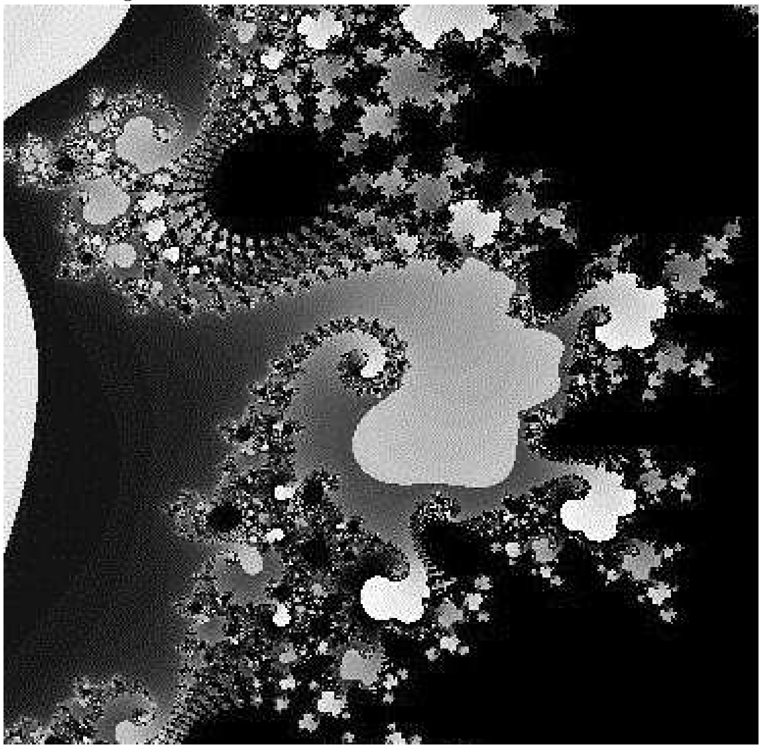
10 The use of complex numbers for your research

- Fractals
  A famous example of a fractal is the Mandelbrot set (see Fig.8), named after Benoit Mandelbrot who discovered it in 1963. The most amazing thing about it is that we can discover an infinitive beauty and complexity in the pictures of fractals although they are generated by a very simple formula. Each point $z = x + iy$ in the complex space is put recursively into

$$z_{n+1} = z_n^2 + c,$$

where $c = x + iy$ is a complex constant. The number of iterations until z reaches a certain value corresponds to the colour at this place of the picture.

Figure 8: picture of a part of the mandelbrot set



- Second order differential equations
  (See 5. in Section 11)
- The strength of electromagnetic fields
  The intensity of an electric field and the intensity of a magnetic field can be described with a complex number. The multiplication rule of complex numbers turns out to have relevance to the physics of the electromagnetic field.
- Almost everything oscillating in physics

11 Further Reading and interesting Web links

1. *'Complex numbers'*, W. Bolton, 1995, Longman Scientific & Technical, Essex, England
2. *'Complex Variables for Mathematics and Engineering'*, John H. Mathews, 1977

3. M. Casco, *Java Applet for adding two complex numbers grafically*, http://www.mcasco.com/oacnav.html
4. Panagiotis Christias, *The Mandelbrot Explorer, Java Applet to zoom into the Mandelbrot set*, http://www.softlab.ece.ntua.gr/mandel/
5. *Homogeneous 2nd Order Euler-Cauchy Equations*, http://www.eng.uml.edu/Dept/Chemical/onlinec/white/math/s2/s2secor/s2secor.html

References

[1] MacTutor History of Mathematics archive, *pictures of Gauß and Euler*, http://www.math.bme.hu/mathhist/

[2] Verena Vanessa Hafner, 1993, *'$\pi = 0$'*, http://www.lake.de/home/lake/sapir/PI.html

[3] Uday Gandbhir, 1995, *'Warum π nicht 0 ist'*, http://www.informatik.uni-konstanz.de/~gandbhir/ (in German)